$K \to \pi\pi$ phenomenology in the presence of electromagnetism

V. Cirigliano[a], J.F. Donoghue[b], E. Golowich[c]

Department of Physics and Astronomy, University of Massachusetts, Amherst, MA 01003, USA

Abstract. We describe the influence of electromagnetism on the phenomenology of $K \to \pi\pi$ decays. This is required because the present data were analyzed without inclusion of electromagnetic radiative corrections, and hence contain several ambiguities and uncertainties which we describe in detail. Our presentation includes a full description of the infrared effects needed for a new experimental analysis. It also describes the general treatment of final state interaction phases, needed because Watson's theorem is no longer valid in the presence of electromagnetism. The phase of the isospin-two amplitude $\mathcal{A}_2$ may be modified by $50 \to 100\%$. We provide a tentative analysis using present data in order to illustrate the sensitivity to electromagnetic effects, and also discuss how the standard treatment of ϵ'/ϵ is modified.

1 Introduction

In this paper we address the effect of electromagnetism on the phenomenology of $K \to \pi\pi$ decays. Our previous work of [1–3] has dealt mainly with the determination of structure dependent EM effects on the $K \to \pi\pi$ amplitudes. It is the aim of the present work to attempt a complete phenomenological analysis. We start by briefly reviewing the standard treatment of $K \to \pi\pi$ decay amplitudes, pointing out that there is room for potentially important isospin breaking effects. We then focus on the effect of electromagnetic interactions (EM); we enumerate the main new features due to EM and describe their impact on the parameterization of the $K \to \pi\pi$ decay amplitudes. Our quantitative analysis begins in Sect. 2 where we take up the problem of the infrared divergences which are induced by electromagnetism. We provide a complete description suitable for use in an experimental analysis in Sect. 3. We also point out that the data on $K \to \pi\pi$ do not lead to a direct extraction of the strong phase shift difference $\delta_0 - \delta_2$, because electromagnetism changes the rescattering phases of the amplitudes. Already a perturbative calculation has provided clues for the presence of sizeable effects [2]. To account for this effect more generally, we provide in Sect. 4 a suitable extension of Watson's theorem, obtained after writing and solving the unitarity constraints in presence of isospin-breaking interactions. Our goal throughout the paper is to relate the theoretical and experimental issues of these decays, in the hope that a future experimental analysis will be undertaken to resolve the substantial experimental uncertainties.

The reason why present experimental information is not adequate is that most of the data was analyzed without the inclusion of electromagnetic radiative corrections. This means that the data in the Particle Data Tables is not fully reliable. Moreover, for certain quantities (namely $\delta_0 - \delta_2$ and the $I = 2$ amplitude A_2) this uncertainty is enhanced by a $\Delta I = 1/2$ enhancement factor of 22. In Sect. 5 we demonstrate these effects by giving an illustrative data analysis, trying our best to interpret the present data. This step is necessarily tentative and likely partially incorrect, as it requires knowledge of the experimental procedure used to deal with soft photons when measuring the branching ratios. In the absence of detailed information from the Particle Data Group (PDG), we adopt the simplest theoretical framework, not necessarily corresponding to the real experimental setup. Within this simple framework we are able to show that the extraction of EM free quantities is quite sensitive to the treatment of infrared photons.

The other interesting phenomenological issue concerns the effect of electromagnetism on CP-violating observables. In Sect. 6 we discuss the impact of our findings on theoretical predictions for ϵ'/ϵ. In particular, we provide a quantitative estimate for the parameter Ω^{EM}, the effect of a $\Delta I = 5/2$ interaction, and the impact of the new rescattering effects on the phase of ϵ'. We conclude the paper with a summary of our findings in Sect. 7.

1.1 Standard phenomenology for $K \to \pi\pi$ decays

Let us start from the conventional phenomenological analysis of the decay amplitudes. There are three physical $K \to \pi\pi$ decay amplitudes[1],

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[a] e-mail: vincenzo@het2.physics.umass.edu
[b] e-mail: donoghue@physics.umass.edu
[c] e-mail: golowich@physics.umass.edu

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[1] The invariant amplitude $\mathcal{A}$ is defined via $_{\mathrm{out}}\langle\pi\pi|K\rangle_{\mathrm{in}} = i(2\pi)^4\delta^{(4)}(p_{\mathrm{out}} - p_{\mathrm{in}})(i\mathcal{A})$

$$\mathcal{A}_{K^0 \to \pi^+ \pi^-} \equiv \mathcal{A}_{+-} \ ,$$
$$\mathcal{A}_{K^0 \to \pi^0 \pi^0} \equiv \mathcal{A}_{00} \ , \qquad \mathcal{A}_{K^+ \to \pi^+ \pi^0} \equiv \mathcal{A}_{+0} \ . \qquad (1)$$

We consider first these amplitudes in the limit of exact isospin symmetry and then identify which modifications must occur in the presence of electromagnetism. In the $I = 0, 2$ two-pion isospin basis, the physical amplitudes are parameterized as:

$$\mathcal{A}_{+-} = A_0 \, e^{i\delta_0} + \sqrt{\frac{1}{2}} A_2 \, e^{i\delta_2} \ ,$$
$$\mathcal{A}_{00} = A_0 \, e^{i\delta_0} - \sqrt{2} A_2 \, e^{i\delta_2} \ , \qquad (2)$$
$$\mathcal{A}_{+0} = \frac{3}{2} A_2 \, e^{i\delta_2} \ .$$

In the above, $A_{0,2}$ are the $\Delta I = 1/2, 3/2$ transition amplitudes corresponding to the $\pi\pi$ final states with isospin equal to $0, 2$ respectively. They are real in the limit of CP conservation. The δ_I are the $I = 0, 2$ $\pi\pi$ scattering S-wave phase shifts at center of mass energy equal to the kaon mass. They enter the parameterization as prescribed by unitarity (the Fermi-Watson theorem). Knowledge of the experimental branching ratios [4] allows one to use (2) to extract the isospin amplitudes. Neglecting the small CP-violation effect, we find[2]

$$A_0 = (5.458 \pm 0.012) \times 10^{-7} M_{K^0} \ ,$$
$$A_2 = (0.2454 \pm 0.0010) \times 10^{-7} M_{K^0} \ , \qquad (3)$$
$$\delta_0 - \delta_2 = (56.7 \pm 3.8)^o \ .$$

Throughout we express the $K \to \pi\pi$ amplitudes in units of $10^{-7} M_{K^0}$, with $M_{K^0} = 0.497672$ GeV.

A careful inspection of these phenomenological results reveals some inconsistencies with other existing pieces of phenomenology and theoretical analysis. These clues seem to suggest that isospin breaking effects (like EM) can play an important role in the phenomenology of $K \to \pi\pi$ decays. The considerations we present below apply to all isospin breaking interactions. In this category one includes both EM and *strong* isospin breaking, produced by the difference in the up and down quark masses. In this work we are concerned exclusively with EM effects (also analyzed in [5–7]). For treatments of strong isospin breaking effects see [8,9].

Isospin breaking interactions will in general mix the amplitudes A_0 and A_2, thus generating potentially big corrections to A_2, proportional to $A_0 \cdot \alpha/\pi$. A related issue concerns the presence of a $\Delta I = 5/2$ component in the interaction. This problem has recently received attention in [10]. A $\Delta I = 5/2$ component will distinguish between the amplitudes A_2 entering in the K^0 and K^+ decays. The expression of A_0, A_2, A_2^+ in terms of $A_{\Delta I}$ is given by:

$$A_0 = A_{1/2} \ ,$$
$$A_2 = A_{3/2} + A_{5/2} \ , \qquad (4)$$
$$A_2^+ = A_{3/2} - 2/3 \, A_{5/2} \ .$$

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[2] Knowledge of the phase difference $\delta_0 - \delta_2$ from other determinations poses the constraint $\cos(\delta_0 - \delta_2) > 0$, implying that $A_0 A_2 > 0$. In this paper, we take A_0 and A_2 as positive numbers

We expect the dominant $\Delta I = 5/2$ effect to arise by combining the large $\Delta I = 1/2$ weak interaction with the $\Delta I = 2$ component of the electromagnetic interaction. A combination of the $\Delta I = 3/2$ hamiltonian with the $\Delta I = 1$ interaction proportional to $m_u - m_d$ is also expected to contribute. However, its effect is expected to be doubly suppressed (by the $\Delta I = 1/2$ rule and the smallness of $m_u - m_d$).

A further problem with the isospin analysis is revealed by looking at the extracted phase shifts. The value $\delta_0 - \delta_2 = (56.7 \pm 3.8)^o$ obtained from kaon decay data has to be compared with information coming from other sectors of low energy phenomenology. In particular, the value extracted from a dispersive treatment of $\pi\pi$ scattering data is $\delta_0 - \delta_2 = (45.2 \pm 1.3^{+4.5}_{-1.6})^o$ [11], and the prediction of ChPT [12] is $\delta_0 - \delta_2 = (45 \pm 6)^o$. These two determinations are mutually compatible. However, there is a sizeable discrepancy with the result obtained in the fit to $K \to \pi\pi$. This can be ascribed to isospin breaking and to a non-vanishing $A_{5/2}$.

The above mentioned issues also affect a proper theoretical understanding of the direct CP-violation parameter ϵ'/ϵ. In particular, the leakage of the octet amplitude in A_2 (due to isospin breaking effects) brings an extra contribution to the CP-violating phase of A_2. In the literature only the leakage due to $m_u - m_d$ isospin breaking effects has been analyzed, and is found to be numerically important [8,9]. Moreover, the presence of a $\Delta I = 5/2$ amplitude introduces an extra term in the usual formulae for ϵ'. Finally, understanding the issue regarding the phase shift $\delta_0 - \delta_2$ will provide a better theoretical determination of the phase of ϵ'.

The above considerations call for a careful analysis of electromagnetic effects on $K \to \pi\pi$ decays.

1.2 Electromagnetism and the $K \to \pi\pi$ amplitudes

One can summarize the effects of electromagnetism on $K \to \pi\pi$ amplitudes as follows:

1. First of all one has to deal with universal infrared (IR) effects, due to photons of long wavelength. These effects are common to all processes with charged external particles and do not depend on details concerning the original interaction. They are represented diagrammatically in Fig. 1, where the dark blob is seen as pointlike by the infrared photons. This class of contributions provides a Coulomb final state interaction (FSI) phase and gives rise to IR divergent amplitudes. Such infrared divergences have to be canceled by considering the effect of soft real photons (Fig. 1b). As we shall show in Sects. 2 and 5, these effects can be described as modifying the phase space factor rather than producing effects on the amplitudes themselves. In order to perform an accurate phenomenological study, it is important to include this class of radiative corrections in the experimental analysis of the branching ratios (see Sect. 5 for details).

2. There are structure dependent effects, sensitive to the form of the original interaction. These are hidden in
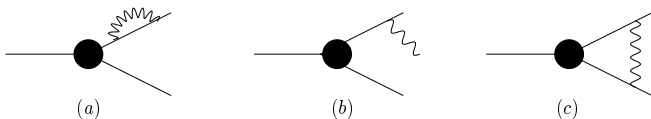
Fig. 1a–c. Electromagnetic contributions relevant for the infrared behavior

Fig. 1 within the large dark vertices. We consider only corrections induced by the dominant (octet) part of the weak hamiltonian. These produce shifts in the isospin amplitudes, and are responsible for possible large contaminations of A_2. They also generate a $\Delta I = 5/2$ amplitude. We have calculated these effects in ChPT and in dispersive matching. [2,3]

3. Finally EM affects the final state interaction. As a consequence the unitarity relations, determining the rescattering phases, are altered. The main modifications are due to the opening of the $\pi\pi + n\gamma$ intermediate channels and the possibility of mixing between two-pion states in isospin $I = 0$ and $I = 2$. These new features imply modifications to the unitarity parameterization, governed by an extension of Watson's theorem that we shall discuss at length in Sect. 4.

Clearly the items enumerated above are intertwined and affect in various respects the way one analyzes the $K \to \pi\pi$ amplitudes. It is important to note that - although being perturbatively small - the new interaction considered breaks the original isospin symmetry on which the parameterization of $K \to \pi\pi$ amplitudes is based. Therefore, to perform a complete analysis of EM effects to $K \to \pi\pi$ decays, one must understand how to parameterize the above mentioned effects.

2 Infrared behavior and isospin amplitudes

2.1 Defining infrared finite amplitudes

Let us start by summarizing the regularization and removal of infrared divergences. These arise in perturbation theory through diagrams in which virtual photons connect external charged legs. The classical works of [13,14] show how to sum the infrared singularities to all orders in perturbation theory and isolate an infrared finite amplitude. We begin by reviewing the content of these works. Let us introduce an IR regulator λ. For our calculation, this takes the form of a photon squared-mass, $\lambda \equiv m_\gamma^2$. Let $\mathcal{A}$ be the amplitude for a generic process involving charged particles. To all orders in the EM interaction $\mathcal{A}$ is given by the expansion

$$\mathcal{A} = \sum_{k=0}^{\infty} \mathcal{A}_k \quad , \tag{5}$$

with $\mathcal{A}_k = \mathcal{O}(\alpha^k)$. Order by order one has the sequence of relations,

$$\mathcal{A}_0 = a_0 \quad ,$$

$$\mathcal{A}_1 = a_0 \cdot \alpha B(\lambda) + a_1 \quad ,$$

$$\mathcal{A}_2 = a_0 \cdot \frac{(\alpha B(\lambda))^2}{2} + a_1 \cdot \alpha B(\lambda) + a_2 \quad , \tag{6}$$

$$\vdots$$

Here $B(\lambda)$ is an infrared divergent function of λ, while the a_k are infrared finite. Summing to all orders results in the compact relation

$$\mathcal{A}(\alpha) = e^{\alpha B(\lambda)} \cdot \sum_{k=0}^{\infty} a_k \equiv e^{\alpha B(\lambda)} \bar{\mathcal{A}}(\alpha) \quad , \tag{7}$$

where all IR-singular dependence appears solely within the exponentiated factor $\alpha B(\lambda)$, which multiplies the infrared finite amplitude $\bar{\mathcal{A}}(\alpha)$. The function $\alpha B(\lambda)$ only depends on the external states and knows nothing on the nature of the interaction generating the process. On the other hand, the amplitude $\bar{\mathcal{A}}(\alpha)$ contains the structure dependent EM effects. These quantities arise naturally in the calculation described in [2], where we provided explicit expressions in the case of $\mathcal{A}_{+-}$.

The construction just described needs to be supplemented by the following comments. As (6) shows, the function $B(\lambda)$ arises as a first order correction in α. This means that a one-loop calculation allows resummation of the IR singularity to all orders. However, the definition of the IR finite part of $B(\lambda)$ is not unique. This means that there is an ambiguity in the way one separates the infrared multiplicative factor from the structure dependent effects. This is a peculiar property of EM radiative corrections and does not affect the definition of physical observables. For example, once one picks a definition for $B(\lambda)$ and follows it throughout the calculation, comparison with experiment will lead to unambiguous extraction of the EM free quantities (like a_0). We will explicitly display our formulas for $B(\lambda)$ below.

The infrared divergences of the amplitudes have now been isolated in an overall factor. Removal of infrared divergences from the expression for the decay rate or cross sections is achieved by taking into account the effect of soft real photons in the external states. This is motivated by the observation that for soft photons, whose energy is below some experimental resolution ω, the generic states n and $n + k\gamma$ cannot be distinguished. The physical observable always involves an inclusive sum over the n and $n + k\gamma$ final states. We shall give details of this in Sect. 5.

2.2 Isospin amplitudes

Having described the construction of IR finite amplitudes, we can now analyze the effect produced on the isospin amplitudes A_0, A_2. We start from the IR finite amplitudes in the charged basis $\{\bar{\mathcal{A}}_{+-}, \bar{\mathcal{A}}_{00}, \bar{\mathcal{A}}_{+0}\}$. It is then possible to define the would-be isospin amplitudes by taking the following linear combinations:

$$\overline{\mathcal{A}}_0 = \frac{2}{3}\overline{\mathcal{A}}_{+-} + \frac{1}{3}\overline{\mathcal{A}}_{00} \quad ,$$

$$\overline{\mathcal{A}}_2 = \frac{\sqrt{2}}{3}(\overline{\mathcal{A}}_{+-} - \overline{\mathcal{A}}_{00}) \ , \qquad (8)$$

$$\overline{\mathcal{A}}_2^+ = \frac{2}{3}\overline{\mathcal{A}}_{+0} \ .$$

The content of (8) is that in the absence of EM and for $m_u = m_d$ the amplitudes $\overline{\mathcal{A}}_I$ truly describe transitions to $\pi\pi$ states with definite isospin. In the presence of isospin breaking interactions, however, the amplitudes $\overline{\mathcal{A}}_I$ become

$$\overline{\mathcal{A}}_I = (A_I + \delta A_I) \, e^{i(\delta_I + \gamma_I)} \ , \qquad (9)$$

with shifts δA_I and γ_I corresponding respectively to the modulus and phase of the original isospin amplitude.

We can summarize the discussion presented so far by displaying the parameterization of the IR finite amplitudes in the charged basis

$$\overline{\mathcal{A}}_{+-} = (A_0 + \delta A_0) \, e^{i(\delta_0 + \gamma_0)} + \frac{1}{\sqrt{2}} (A_2 + \delta A_2) \, e^{i(\delta_2 + \gamma_2)} \ ,$$

$$\overline{\mathcal{A}}_{00} = (A_0 + \delta A_0) \, e^{i(\delta_0 + \gamma_0)} - \sqrt{2} (A_2 + \delta A_2) \, e^{i(\delta_2 + \gamma_2)} \ ,$$

$$\overline{\mathcal{A}}_{+0} = \frac{3}{2} (A_2 + \delta A_2^+) \, e^{i(\delta_2 + \gamma_2^+)} \ , \qquad (10)$$

where δA_I and γ_I contain IR finite EM effects as well as other isospin breaking terms. This parameterization has to be compared with the isospin invariant expressions in (2). We recall here that this parameterization holds for the IR finite amplitude as defined in (7). We also observe that the shifts δA_2^+ and γ_2^+ in $\overline{\mathcal{A}}_{+0}$ are distinct from the corresponding shifts in $\overline{\mathcal{A}}_{+-}$ and $\overline{\mathcal{A}}_{00}$, as a consequence of the $\Delta I = 5/2$ amplitude induced by electromagnetism. In the notation of Eq. (4) one has:

$$\delta A_0 = \delta A_{1/2} \ ,$$
$$\delta A_2 = \delta A_{3/2} + A_{5/2} \ , \qquad (11)$$
$$\delta A_2^+ = \delta A_{3/2} - 2/3 \, A_{5/2} \ .$$

2.3 Discussion

The parameterization given in (10),(11) provides the basis for any phenomenological analysis of $K \to \pi\pi$ decays with inclusion of isospin breaking (due to strong and electromagnetic effects). Comparison with experimental branching ratios (see next Section for some caveats related to radiative corrections) allows one to arrive at relations among the different parameters. Examples of such analysis are given in [10] and the third paper of [9].

It is legitimate at this point to ask what do we know about the parameters entering (10) and what do we want to extract from the comparison with experiment.

1. First, let us consider the amplitudes A_0 and A_2. Many theoretical efforts have been devoted to their calculation, and no calculation can be considered fully satisfactory at present. We want to extract these parameters from the comparison with the branching ratios, eliminating the electromagnetic isospin breaking contaminations.

2. Next, one has the shifts δA_0, δA_2, and δA_2^+. We have calculated the EM contributions to them [2,3], and we are quite confident that our results capture the true underlying physics within the theoretical uncertainty quoted. In particular, our results provide an estimate for $A_{5/2}$. Explicitly we find [3]:

$$\delta A_0^{\text{EM}} = (0.0253 \pm 0.0072) \cdot 10^{-7} \, M_{K^0} \ ,$$
$$\delta A_2^{\text{EM}} = (0.0147 \pm 0.0063) \cdot 10^{-7} \, M_{K^0} \ ,$$
$$\delta A_2^{+\ \text{EM}} = (0.008 \pm 0.0088) \cdot 10^{-7} \, M_{K^0} \ ,$$
$$A_{5/2}^{\text{EM}} = (0.0137 \pm 0.0097) \cdot 10^{-7} \, M_{K^0} \ .$$

For recent estimates of the size of non-EM isospin breaking effects, see [9]. The corrections $\delta A_{\Delta I}^{\text{iso-brk}}$ contain only a negligible $\Delta I = 5/2$ component, as the only source for it would be a combination of the $\Delta I = 1$ interaction proportional to $(m_u - m_d)$ with the suppressed $\Delta I = 3/2$ weak interaction. This ensures that $\delta A_{1/2,3/2}^{\text{iso-brk}}$ can be reabsorbed into $A_{0,2}$. Therefore, in what follows A_0 and A_2 still contain strong isospin breaking contaminations. These can be subtracted by using the results of [9].

3. Finally, let us consider the rescattering phases. In absence of isospin-breaking the phases γ_I vanish as a consequence of Watson's theorem. The strong phases δ_I at $s = M_K^2$ are known through dispersive treatment of $\pi\pi$ scattering data[3]. The inclusion of strong isospin breaking effects still gives $\gamma_I = 0$, to first order in $m_u - m_d$ [10]. It is the inclusion of EM corrections that generates nonzero γ_I. Since the phase $\delta_2 + \gamma_2^+$ does not enter any physical relation, we disregard it from now on. As for the phases γ_0 and γ_2, we can relate them to EM effects in the final state interaction. This can be done in perturbation theory [2] or in a more general setting provided by unitarity. As we shall discuss in Sect. 4, our present knowledge of $\gamma_{0,2}$ reveals a large value of γ_2 and relies on a lowest order analysis of EM corrections to $\pi\pi$ scattering. Next to leading order corrections can be quite large at $s = M_K^2$, and this adds substantial uncertainty to γ_2.

3 Analysis in the presence of electromagnetism

The CP-conserving sector of $K \to \pi\pi$ phenomenology relies essentially on three experimental numbers. These are the partial decay widths of K_S into $\pi^+\pi^-$, $\pi^0\pi^0$ and of $K^\pm$ into $\pi^\pm\pi^0$. Knowledge of these numbers allows one to extract the invariant decay amplitudes and to compare with theoretical calculations. In this section we describe the procedure to be followed in order to extract the invariant amplitudes in presence of radiative corrections.

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[3] For a list of references see [10]

3.1 The fitting procedure in presence of radiative corrections

In presence of radiative corrections, the appropriate expression to be used when comparing theory and experiment is:

$$\Gamma_n(\omega) \;=\; \frac{\Phi_n}{2\sqrt{s_n}} \, |\overline{\mathcal{A}}_n|^2 \, G_n(\omega) \quad, \qquad (12)$$

where $n = \{+-, 00, +0\}$. In (12) the left hand side Γ_n represents the measured partial width and we indicate with the parameter ω its dependence on the way soft photons are treated in the data analysis. It is clear that the use of different cuts leads to different values for the decay widths, because of the inclusion of different portions of the corresponding radiative channel $(n + \gamma)$ in the data sample. On the right hand side of Eq. (12) one has the kinematical parameters s_n (squared center-of-mass energy of the process) and Φ_n (the two body invariant phase space associated with the decay). The quantities related to the dynamics are $\overline{\mathcal{A}}_n$ and $G_n(\omega)$. $\overline{\mathcal{A}}_n$ is the infrared finite invariant amplitude, as defined in Sect. 2, (10). It contains the true weak transition component that we wish to ultimately extract as well as infrared finite electromagnetic corrections. $G_n(\omega)$ is the infrared factor associated with the combined effect of virtual and real photons. The latter contribution involves an integration over the soft-photon phase space: this has to be done with the *same* prescription used in extracting the experimental number $\Gamma_n(\omega)$.

The extraction of IR-free invariant amplitudes is a straightforward consequence of (12). Specializing to the $K \to \pi\pi$ case, one has:

$$\frac{|\overline{\mathcal{A}}_{+-}|}{M_{K^0}} \;=\; \frac{1}{\sqrt{2}} \sqrt{\frac{8\pi}{p_{+-}} \frac{\Gamma_{+-}(\omega)}{G_{+-}(\omega)}} \;,$$

$$\frac{|\overline{\mathcal{A}}_{00}|}{M_{K^0}} \;=\; \sqrt{\frac{8\pi}{p_{00}} \Gamma_{00}} \;, \qquad (13)$$

$$\frac{|\overline{\mathcal{A}}_{+0}|}{M_{K^+}} \;=\; \sqrt{\frac{8\pi}{p_{+0}} \frac{\Gamma_{+0}(\omega)}{G_{+0}(\omega)}} \;,$$

where

$$p_{+-} = \sqrt{\left(\frac{M_{K^0}}{2}\right)^2 - M_{\pi^+}^2} \;,$$

$$p_{00} = \sqrt{\left(\frac{M_{K^0}}{2}\right)^2 - M_{\pi^0}^2} \;, \qquad (14)$$

$$p_{+0} = \sqrt{\left(\frac{M_{K^+}}{2} + \frac{M_{\pi^0}^2 - M_{\pi^+}^2}{2M_{K^+}}\right)^2 - M_{\pi^0}^2} \;.$$

In order to carry out the procedure one needs the physical masses, the experimental input $\Gamma_n(\omega)$, and the corresponding theoretical infrared parameters $G_{+-,+0}(\omega)$. If things are done properly, the ω dependence cancels in the ratios on the right hand side of (13), which provides then the values of $|\overline{\mathcal{A}}_n|$. Having the amplitudes $|\overline{\mathcal{A}}_n|$ one can

then use (10) to obtain relations between the physical parameters. However, there is an important issue to be addressed in order to complete the program outlined above: it is the proper definition of the infrared factors $G_{+-}(\omega)$ and $G_{+0}(\omega)$, to which we now turn.

3.2 The infrared factors

As already noted, the theoretical definition of $G_{+-}(\omega)$ and $G_{+0}(\omega)$ involves integration over the soft-photon phase space, with cuts generically indicated by ω. The infrared factor *is* experiment dependent and accounts for the component of the radiative mode ($\pi\pi\gamma$ in our case) included in the parent mode ($\pi\pi$) branching ratio. We shall work to order α and thus include only the effect of the $\pi\pi\gamma$ radiative state. While we display the ingredients needed for any experimental analysis, we carry the treatment to completion for the case where the infrared sensitivity is isotropic in the center of mass. That is, we display the explicit formulas obtained when one integrates over the $\pi\pi\gamma$ phase space applying an isotropic cutoff ω on the photon energy E_γ in the center of mass frame. This is most natural to use in the data analysis for an experiment such as KLOE at DAΦNE, given the working conditions of the machine and the detector geometry. A more detailed study (either theoretical or experimental) is required in order to apply radiative corrections to the data analysis of other experiments. The presence of several high statistics experiments offers a unique opportunity for performing an accurate measurement of $K \to \pi\pi$ branching ratios, including the effect of radiative corrections. Such an analysis would fill the present gap in the study of radiative corrections to $K^0 \to \pi\pi$, and would therefore be highly desirable.

Calculation of the factors $G_{+-}(\omega)$ and $G_{+0}(\omega)$ requires consideration of a combination of effects due to virtual photons in the amplitudes $\mathcal{A}_{+-,+0}$ (field theoretic version of the non-relativistic Gamow factors) and to soft real photons entering the process $K \to \pi\pi + \gamma$. Moreover, in the infrared region of the spectrum, the amplitude for $K \to \pi\pi + \gamma$ is dominated by the internal bremsstrahlung (IB) component, proportional to the non-radiative amplitude. For the case in question one has:

$$\mathcal{A}_{+-\gamma}^{IB} = e\mathcal{A}_{+-} \left(\frac{\epsilon \cdot p_+}{q \cdot p_+} - \frac{\epsilon \cdot p_-}{q \cdot p_-}\right) \;,$$

$$\mathcal{A}_{+0\gamma}^{IB} = e\mathcal{A}_{+0} \left(\frac{\epsilon \cdot p_+}{q \cdot p_+} - \frac{\epsilon \cdot p_K}{q \cdot p_K}\right) \;, \qquad (15)$$

where ϵ and q are the polarization and momentum of the emitted photon. Now, the infrared-finite observable decay rate is

$$\Gamma_n(\omega) = \Gamma_n \;+\; \Gamma_{n\gamma}(\omega) \quad , \qquad (16)$$

where $n = +-, +0$ and

$$\Gamma_n = \frac{1}{2M_K} \int d\Phi_n \left| \overline{\mathcal{A}}_n \left(1 + \alpha B_n(m_\gamma)\right) \right|^2 \;, (17)$$

$$\Gamma_{n\gamma}(\omega) = \frac{1}{2M_K} \int_{E_\gamma < \omega} d\Phi_{n\gamma} \left| \mathcal{A}_{n\gamma} \right|^2$$

$$= \frac{1}{2M_K} \int d\Phi_n \left|\overline{\mathcal{A}}_n\right|^2 \, I_n(m_\gamma, \omega) \quad . \qquad (18)$$

At this stage the rest of the calculation becomes dependent on the geometry of the experiment. For a kaon at rest with an isotropic detector, the acceptance cutoff $E_\gamma < \omega$ will be the same in all directions. For a kaon in flight, the acceptance may be different for photons emitted in different directions. In this latter situation, the integral in Eq. (18) needs to be numerically integrated over the detector acceptance and then added to the two body result, Eq. (17). This will then be finite in the limit of $m_\gamma \to 0$, and will allow the measurement of the IR finite amplitude $\overline{\mathcal{A}}_n$. We carry out this procedure explicitly below for the case of an isotropic cutoff.

In (17),(18) $d\Phi_k$ is the differential phase space factor for each process, $\overline{\mathcal{A}}_n$ is the IR finite amplitude, as defined in Sect. 2 by extracting the IR divergent functions B_n. The B_n can be calculated by considering one-loop diagrams with virtual photons connecting the external legs and using a point-like vertex for the weak interaction. The definition of B_n is not unique due to the possibility of adding IR and UV finite terms to it. The explicit expressions given below[4] fully specify our choice of what goes into B_n and what goes into the structure dependent shifts δA_n. The second expression in Eq. (18) describes the factorization of the $\pi\pi$ and γ phase spaces, valid with an accuracy of ω/M_K. Explicitly one has:

$$I_n(m_\gamma, \omega) = \int_{E_\gamma < \omega} \frac{d^3q}{(2\pi)^3 2E_\gamma} \sum_{pol} \left|\frac{\mathcal{A}_{n\gamma}^{IB}}{\mathcal{A}_n}\right|^2 \quad . \qquad (19)$$

Combining the above results one arrives at

$$G_n(\omega) = 1 + 2\alpha \, \mathcal{R}e B_n(m_\gamma) + I_n(m_\gamma, \omega) + \mathcal{O}(\alpha^2) \quad . \quad (20)$$

We now collect the explicit form of the functions $B_{+-,+0}$ and $I_{+-,+0}$, entering in the definition of $G_{+-,+0}$ (see 20). For $G_{+-}(\omega)$ one needs:

$$B_{+-}(m_\gamma^2) = \frac{1}{4\pi} \left[2a(\beta) \ln \frac{M_\pi^2}{m_\gamma^2} + H_{+-}(\beta) \right.$$
$$\left. + i\pi \left(\frac{1+\beta^2}{\beta} \ln \frac{M_K^2 \beta^2}{m_\gamma^2} - \beta \right) \right] \quad ,$$
$$I_{+-}(m_\gamma, \omega) = \frac{\alpha}{\pi} \left[a(\beta) \ln \left(\frac{m_\gamma}{2\omega}\right)^2 + F_{+-}(\beta) \right] \quad , \quad (21)$$

where

$$\beta = (1 - 4M_\pi^2/M_K^2)^{1/2} \qquad (22)$$

and

$$a(\beta) = 1 + \frac{1+\beta^2}{2\beta} \ln \left(\frac{1-\beta}{1+\beta}\right) \quad ,$$
$$H_{+-}(\beta) = \frac{1+\beta^2}{2\beta} \left[\pi^2 + \ln \frac{1+\beta}{1-\beta} \ln \frac{1-\beta^2}{4\beta^2} \right.$$
$$\left. + 2f\left(\frac{1+\beta}{2\beta}\right) - 2f\left(\frac{\beta-1}{2\beta}\right) \right]$$

---
[4] See also [2]

$$+ 2 + \beta \ln \frac{1+\beta}{1-\beta} \quad ,$$
$$F_{+-}(\beta) = \frac{1}{\beta} \ln \frac{1+\beta}{1-\beta} + \frac{1+\beta^2}{2\beta} \left[2f(-\beta) - 2f(\beta) \right.$$
$$+ f\left(\frac{1+\beta}{2}\right) - f\left(\frac{1-\beta}{2}\right) + \frac{1}{2} \ln \frac{1+\beta}{1-\beta}$$
$$\times \ln(1-\beta^2) + \ln 2 \, \ln \frac{1-\beta}{1+\beta} \bigg]$$
$$f(x) = -\int_0^x dt \, \frac{1}{t} \ln|1-t| \quad . \qquad (23)$$

We note that B_{+-} includes the Coulomb factor $\pi\alpha/v_{\rm rel}$ (first term in $H_{+-}(\beta)$), as well as typical field theoretic effects. As for $G_{+0}(\omega)$, one has:

$$B_{+0}(m_\gamma^2) = \frac{1}{4\pi} \left[2b(\beta) \ln \frac{M_\pi^2}{m_\gamma^2} + H_{+0}(\beta) \right] \quad ,$$
$$I_{+0}(m_\gamma, \omega) = \frac{\alpha}{\pi} \left[b(\beta) \ln \left(\frac{m_\gamma}{2\omega}\right)^2 + F_{+0}(\beta) \right] \quad , \quad (24)$$

where

$$b(\beta) = 1 + \frac{1}{2\beta} \log \left(\frac{1-\beta}{1+\beta}\right) \quad ,$$
$$H_{+0}(\beta) = -\frac{1}{\beta} \left[\frac{1}{2} \log \left(\frac{1+\beta}{2}\right) \log \left(\frac{2+2\beta}{(1-\beta)^2}\right) \right.$$
$$- \frac{1}{2} \log \left(\frac{1-\beta}{2}\right) \log \left(\frac{2-2\beta}{(1+\beta)^2}\right)$$
$$+ \log \left(\frac{4\beta}{1-\beta^2}\right) \log \left(\frac{1+\beta}{1-\beta}\right) + f\left(\frac{1+\beta}{2\beta}\right)$$
$$- f\left(\frac{\beta-1}{2\beta}\right) + f\left(-\frac{(1-\beta)^2}{4\beta}\right)$$
$$\left. - f\left(\frac{(1+\beta)^2}{4\beta}\right) \right] + 2 + \log \frac{1-\beta^2}{4} - \frac{2}{1-\beta}$$
$$\times \log \left(\frac{1+\beta}{2}\right) - \frac{2}{1+\beta} \log \left(\frac{1-\beta}{2}\right)$$

and

$$F_{+0}(\beta) = 1 + \frac{1}{2\beta} \log \left(\frac{1+\beta}{1-\beta}\right) - \frac{4}{1-\beta^2}$$
$$\times \int_{-1}^{+1} dx \frac{E(x)}{D(x)p(x)} \log \left(\frac{E(x)+p(x)}{E(x)-p(x)}\right)$$
$$D(x) = (x - x_1)(x - x_2)$$
$$x_{1/2} = \frac{M_K^2}{M_\pi^2}(1 \pm \beta) - 1$$
$$E(x) = \frac{M_K}{4}(3 - x)$$
$$p(x) = \frac{M_K}{4}\beta(1 + x) \quad . \qquad (25)$$

G_{+-} and G_{+0} do not depend on the infrared regulator m_γ. We display plots of the functions $G_{+-}(\omega)$ and $G_{+0}(\omega)$ on a typical range of values for the parameter ω
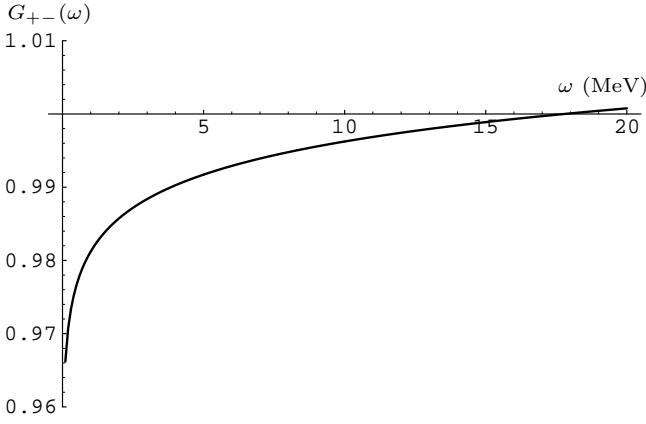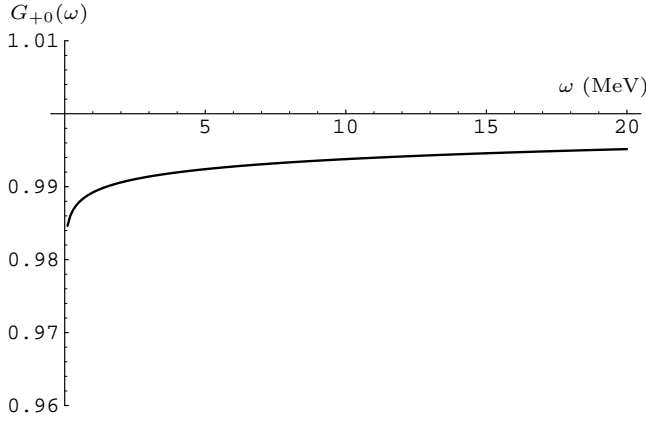
Fig. 2. The function $G_{+-}(\omega)$

Fig. 3. The function $G_{+0}(\omega)$

in Figs. 2 and 3. The functions $G_n(\omega)$ are the only EM effects previously considered in the literature [5], although with a slightly different definition. In fact, the works of [5] use a point-like vertex for the weak interaction, and therefore are not sensitive to structure dependent corrections. In these works the EM effects due to wavefunction renormalization and vertex correction go entirely in the definition of B_n (this includes also the UV divergent terms, regulated by means of a cutoff). Apart from the cutoff-dependent term and an extra finite contribution, our expressions match the ones given in the second and third papers of [5].

4 Effect of EM on $K \to \pi\pi$ phases

The amplitude parameterization we have used for the previous analysis already implies that $K \to \pi\pi$ data do not provide direct information on the strong $\pi\pi$ phase shift difference $\delta_0 - \delta_2$. This would only be true in the isospin limit ($\gamma_I = 0$). In this limit, the requirement that strong interaction phases appear in the weak decays is known as Watson's theorem, which is valid whenever the final state rescattering involves only elastic scattering. Despite the fact that isospin breaking has been long understood to cause mixing of weak amplitudes, there has been no

recognition that the strong interaction phases no longer suffice to describe the rescattering effects. This occurs because elastic rescattering is no longer the full content of final state interaction, so that the conditions for the application of Watson's theorem no longer apply. Moreover, there exists a sizeable discrepancy between the determination of $\delta_0 - \delta_2$ from $K \to \pi\pi$ data using isospin relations and the favored value of the phase shift difference known by other determinations. This seems to point to violations of Watson's theorem. It is our purpose to set the framework for the correct treatment of this problem in the isospin breaking real world. We shall accomplish this by writing a coupled channel unitarity constraint in the presence of EM interactions (and isospin breaking in general) and solving for the parameters γ_0 and γ_2 entering (10). This analysis will complement and extend the perturbative results obtained in [2], which already indicated a large value of γ_2. We defer to the next section the extraction of $\delta_0 - \delta_2$ and the uncertainty to be associated with it.

4.1 Extended unitarity relations

The first step in our program is writing down meaningful unitarity relations in the presence of EM interactions. Here, it is natural to work in the *charged* basis $\{\pi^+\pi^-, \pi^0\pi^0\}$ and then try to recover the notion of isospin amplitudes. In order to fix the notation, let us start from the unitarity relations involving the decay amplitudes of K^0 to $\{\pi^+\pi^-, \pi^0\pi^0\}$ in the limit in which EM is turned off. Then, only the $\pi\pi$ intermediate states have to be taken into account and one finds:

$$\mathcal{A}_{+-} - \mathcal{A}^*_{+-} = i\left(\mathcal{T}^*_{+-;+-} \times \mathcal{A}_{+-} + \mathcal{T}^*_{00;+-} \times \mathcal{A}_{00}\right) ,$$
$$\mathcal{A}_{00} - \mathcal{A}^*_{00} = i\left(\mathcal{T}^*_{+-;00} \times \mathcal{A}_{+-} + \mathcal{T}^*_{00;00} \times \mathcal{A}_{00}\right) , \quad (26)$$

In (26) $\mathcal{A}_{+-}$ and $\mathcal{A}_{00}$ represent the K^0 decay amplitudes and $\mathcal{T}_{f;i}$ is the T-matrix element for the transition $i \to f$ (in this case it only involves pion-pion scattering). '$\mathcal{T}^* \times \mathcal{A}$' denotes the product of amplitudes integrated over the intermediate state phase space. In the case considered here of two-pion intermediate states, one has:

$$\mathcal{T}^* \times \mathcal{A} \equiv \int d\Phi_2 \, \mathcal{T}^* \mathcal{A} = \Phi_s \, 4\beta \, \mathcal{A} \cdot \mathcal{T}^* , \quad (27)$$

where $\beta = (1 - 4m_\pi^2/m_K^2)^{-1/2}$ is the pion velocity in the kaon rest frame. Φ_s is the symmetry factor for identical particles (equal to $1/2$ for the $\pi^0\pi^0$ state) and $\mathcal{T}$ is the S-wave projection of the $\pi\pi$ scattering amplitude $T(\cos\theta)$, defined by:

$$\mathcal{T} = \frac{1}{64\pi} \int_{-1}^{+1} d(\cos\theta) \, T(\cos\theta) . \quad (28)$$

Turning on EM interactions introduces isospin breaking dynamics as well as IR singularities in the amplitudes and the opening of intermediate radiative channels. Specifically, $\mathcal{A}_{+-}$, $\mathcal{T}_{00,+-}$, and $\mathcal{T}_{+-,+-}$ become IR divergent

and $\mathcal{T}_{+-,+-}$ acquires a purely Coulomb component (also IR singular). The work of [13,14], summarized in Sect. 2 teaches us that one can always isolate the singularity in a multiplicative exponential factor

$$\mathcal{A}_{f,i} = e^{\alpha B_{f,i}} \, \overline{\mathcal{A}}_{f,i} \; . \tag{29}$$

Here $\overline{\mathcal{A}}_{f,i}$ is the IR finite amplitude and $B_{f,i}(m_\gamma)$ is the IR singular factor that depends only on the external states. We shall only need the factor B_{+-}, already encountered in this paper, associated with a pair of charged pions in the initial or final state. We note that $B_{f,i}(m_\gamma)$ is in general complex. In particular, its imaginary part is equal to the Coulomb scattering phase shift associated with each pair of charged particles in the initial and final states [14].

Upon integrating over the phase space and using the above mentioned property on the Coulomb phases, one can rewrite (26) in terms of IR finite quantities and $B_{f,i}$ factors. Moreover, due to the integration over the phase space some contributions in the $B_{f,i}$ factors simplify and one ends up with:

$$\overline{\mathcal{A}}_{+-} - \overline{\mathcal{A}}^*_{+-}$$
$$= i \left(\overline{\mathcal{T}}^*_{+-;+-} \times \overline{\mathcal{A}}_{+-} \, e^{2\alpha \mathcal{R}e B_{+-}} + \overline{\mathcal{T}}^*_{00;+-} \times \overline{\mathcal{A}}_{00} \right) \; ,$$
$$\overline{\mathcal{A}}_{00} - \overline{\mathcal{A}}^*_{00}$$
$$= i \left(\overline{\mathcal{T}}^*_{+-;00} \times \overline{\mathcal{A}}_{+-} \, e^{2\alpha \mathcal{R}e B_{+-}} + \overline{\mathcal{T}}^*_{00;00} \times \overline{\mathcal{A}}_{00} \right) \; . \tag{30}$$

Note that now $\overline{\mathcal{T}}_{+-,+-}$ is the IR finite $\pi^+\pi^- \to \pi^+\pi^-$ amplitude subtracted of its purely Coulomb term.

Equation (30) contains IR singularities, but the analysis is still missing an important effect of EM: the opening of inelastic radiative channels. This is the key ingredient in obtaining an IR finite set of unitarity constraints, as it was in obtaining an IR finite cross section or decay rate. In fact, the IR singularities will cancel in the sum over the $\pi^+\pi^-$ and $\pi^+\pi^-\gamma$ intermediate states, with the same mechanism described in the definition of Γ_{+-}, Γ_{+0} earlier in Sect. 3. Working at order $\mathcal{O}(\alpha)$, we consider only the radiative state $\pi^+\pi^-\gamma$. For our analysis we require the amplitudes for $K^0 \to \pi^+\pi^-\gamma$ and $\pi\pi \to \pi^+\pi^-\gamma$. We include only the internal bremsstrahlung component of these amplitudes, known to be dominant over possible direct emission terms. Now one has to integrate over the full $\pi^+\pi^-\gamma$ phase space and the final result for the unitarity condition reads:

$$\mathcal{I}m \begin{pmatrix} \overline{\mathcal{A}}_{+-} \\ \overline{\mathcal{A}}_{00} \end{pmatrix} = \beta \begin{pmatrix} 2\,\overline{\mathcal{T}}^*_{+-;+-}\,(1+\Delta_{+-}) & \overline{\mathcal{T}}^*_{00;+-} \\ 2\,\overline{\mathcal{T}}^*_{+-;00}\,(1+\Delta_{+-}) & \overline{\mathcal{T}}^*_{00;00} \end{pmatrix}$$
$$\times \begin{pmatrix} \overline{\mathcal{A}}_{+-} \\ \overline{\mathcal{A}}_{00} \end{pmatrix} \; . \tag{31}$$

We recall that $\overline{\mathcal{T}}_{a,b}$ are the S-wave projections of the $\pi\pi$ scattering matrix. Δ_{+-} is the IR finite remnant of the sum of IR singular terms in the $\pi^+\pi^-$ and $\pi^+\pi^-\gamma$ intermediate states. In terms of the notation of Sect. 3.2, it is given by:

$$\Delta_{+-} = -\frac{2\delta M_\pi^2}{\beta^2 M_K^2} + 2\alpha \mathcal{R}e B_{+-}(m_\gamma) + e^2 \frac{1}{\Phi_{+-}} \int d\Phi_{+-\gamma}$$

$$\times \sum_{pol} \left| \frac{q_+ \cdot \epsilon}{q_+ \cdot k} - \frac{q_- \cdot \epsilon}{q_- \cdot k} \right|^2 \; . \tag{32}$$

Here the first term is the phase space correction due to the EM mass-shift of charged pions. The second term is the effect of infrared virtual photons, while the third term is the effect of real soft photons in the intermediate state $\pi^+\pi^-\gamma$. Numerically we find (displaying separately the phase space contribution and the remainder):

$$\Delta_{+-} = (-14.8 + 10.8) \cdot 10^{-3} = -4.0 \cdot 10^{-3} \; . \tag{33}$$

4.2 From charge to isospin basis

Assuming unitarity of the S matrix, we have thus far obtained a set of relations containing the IR finite amplitudes in the charge basis. In order to compare with usual treatments of this problem, we rotate now to the isospin basis for the $K \to \pi\pi$ amplitudes,

$$\overline{\mathcal{A}}_{\text{ISO}} = \begin{pmatrix} \overline{\mathcal{A}}_0 \\ \overline{\mathcal{A}}_2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ \sqrt{2} & -\sqrt{2} \end{pmatrix} \begin{pmatrix} \overline{\mathcal{A}}_{+-} \\ \overline{\mathcal{A}}_{00} \end{pmatrix} \; . \tag{34}$$

Applying the same transformation to the whole system in Eq. (31) one obtains in matrix form:

$$\mathcal{I}m\overline{\mathcal{A}}_{\text{ISO}} = \beta \left(\overline{\mathcal{T}}^\dagger_{\text{ISO}} + \mathcal{R} \right) \overline{\mathcal{A}}_{\text{ISO}} \; , \tag{35}$$

where

$$\overline{\mathcal{T}}_{\text{ISO}} = \begin{pmatrix} T_0 & T_{02} \\ T_{20} & T_2 \end{pmatrix} \; , \tag{36}$$

and

$$\mathcal{R} = \begin{pmatrix} R_{00} & R_{02} \\ R_{20} & R_{22} \end{pmatrix} = \frac{1}{3} \Delta_{+-} \begin{pmatrix} 2T_0^* & \sqrt{2}T_0^* \\ \sqrt{2}T_2^* & T_2^* \end{pmatrix} \; . \tag{37}$$

$\overline{\mathcal{T}}_{\text{ISO}}$ is the $\pi\pi$ scattering matrix in the isospin basis, while the matrix $\mathcal{R}$, proportional to Δ_{+-}, contains the effect of IR radiative corrections and the radiative intermediate channel. The $\pi\pi$ scattering T-matrix now involves both strong and EM interactions, and thus contains isospin-violating matrix elements. In the conventions used in our work, the amplitudes for the $\pi\pi$ scattering in the isospin basis are expressed in terms of the charged ones as:

$$T_0 = \frac{1}{3} \left(4\overline{\mathcal{T}}_{+-,+-} + \overline{\mathcal{T}}_{00,00} + 4\overline{\mathcal{T}}_{00,+-} \right) \; ,$$
$$T_2 = \frac{2}{3} \left(\overline{\mathcal{T}}_{+-,+-} + \overline{\mathcal{T}}_{00,00} - 2\overline{\mathcal{T}}_{00,+-} \right) \; , \tag{38}$$
$$T_{20} = T_{02} = \frac{\sqrt{2}}{3} \left(2\overline{\mathcal{T}}_{+-,+-} - \overline{\mathcal{T}}_{00,00} - \overline{\mathcal{T}}_{00,+-} \right) \; .$$

A general parameterization of the $\pi\pi$ transition matrix in the isospin basis is:

$$\beta \overline{\mathcal{T}}_{\text{ISO}} = \begin{pmatrix} (\eta_0 \, e^{2i\delta_0} - 1)/(2i) & a e^{i(\delta_0 + \delta_2 + \Delta)} \\ a e^{i(\delta_0 + \delta_2 + \Delta)} & (\eta_2 \, e^{2i\delta_2} - 1)/(2i) \end{pmatrix} \; . \tag{39}$$

In this parameterization we allow for isospin mixing (the off-diagonal parameter a) and for possible non-unitarity in the $\pi\pi$ two dimensional subspace (due to opening of other channels). This is accomplished by introducing the inelasticity parameters $\eta_{0,2}$ and the extra phase Δ in the off-diagonal term. The parameters η_I are of order $1 + \mathcal{O}(\alpha^2)$, while a and Δ are of order α.

The form given in (39) is fully general and includes all isospin breaking effects. However, strong isospin breaking is expected to induce only subleading rescattering effects. In fact, $m_u - m_d \neq 0$ produces an $I = 1$ perturbation to the original interaction. This is not sufficient to mix the $I = 0$ and $I = 2$ $\pi\pi$ scattering states when treated to first order, nor does it cause a splitting of the masses of the charged and neutral pions. This implies that elastic scattering of these states is still the only option, and to first order in $m_u - m_d$ the parameter a does not receive contributions. The values of the phases δ_I may in principle be slightly modified by the quark mass effect, yet this is contained in the measured values of the experimental phase shifts.

4.3 Solution for $\gamma_{0,2}$

We are now in position to explore the consequences of unitarity on the rescattering phases $\gamma_{0,2}$. We write

$$\overline{\mathcal{A}}_I = \overline{A}_I \, e^{i(\delta_I + \gamma_I)} \tag{40}$$

and insert these expressions into Eq. (35). We then solve for $\sin\gamma_0$ and $\sin\gamma_2$ to first order in α, taking into account the $\Delta I = 1/2$ hierarchy of magnitudes. After some simple algebra, we obtain the solutions

$$\sin\gamma_0 = \beta \left(\mathcal{R}e R_{00} - \tan\delta_0 \, \mathcal{I}m R_{00} \right) \simeq \mathcal{O}(\alpha \sin\delta_0)$$

$$\sin\gamma_2 = \beta \frac{\overline{A}_0}{\overline{A}_2} \left[T_{20} + \frac{1}{\cos\delta_2} \left(\mathcal{R}e R_{20} \cos\delta_0 \right. \right.$$

$$\left. \left. - \mathcal{I}m R_{20} \sin\delta_0 \right) \right] \tag{41}$$

The most important feature of these results is the factor $\overline{A}_0/\overline{A}_2$ in the formula for $\sin\gamma_2$. This implies that even though the non-elastic scattering is electromagnetic in origin, it is enhanced by a large factor that allows the net change to be significant. Equation (41) gives us the desired expression relating the phase γ_2 to isospin breaking rescattering effects. These are contained in the parameters T_{20}, the mixing amplitude between $\pi\pi$ states, and R_{20}. This last parameter contains the effect of the radiative intermediate channel $\pi^+\pi^-\gamma$ as well as the phase space correction. We note that (41) is a generalization of the relation obtained at one loop in ChPT. However, inspection reveals that the perturbative determination contains only the phase space effect and the T_{20} mixing in lowest order.

In attempting to estimate the magnitudes of the new phases, we are hampered by the fact that the analysis of electromagnetic effects in $\pi\pi$ scattering is not yet complete in the literature. Two groups have provided analyses of reactions involving neutral mesons [16], but the channels with all charged particles are not yet fully analyzed. We require the scattering elements at center-of-mass energy equal to the kaon mass. The threshold matrix elements are known from simple tree level calculations, and we will use these in our estimate below. However, the amplitudes can experience large changes at $s = M_K^2$, and one needs at least one-loop chiral perturbation theory in order to obtain these. As these results become available, they can be used to update our numerical estimates.

We estimate the off diagonal parameter at lowest order in chiral symmetry obtaining:

$$T_{02} = \frac{\sqrt{2}}{3} \cdot \frac{\delta M_\pi^2}{8\pi F_\pi^2} \simeq 2.7 \times 10^{-3} \; . \tag{42}$$

For the parameter R_{20}, proportional to the radiative effect, one has

$$R_{20} = \frac{\sqrt{2}}{3} \Delta_{+-} T_2^* \; , \tag{43}$$

and we use the form

$$T_2 = \frac{1}{\beta} \, e^{i\delta_2} \, \sin\delta_2 \; , \tag{44}$$

with the phenomenological central value of $\delta_2 = -7.8^o$ [11]. Numerically this leads to

$$\mathcal{R}e R_{20} = 0.280 \cdot 10^{-3}$$
$$\mathcal{I}m R_{20} = 0.034 \cdot 10^{-3} \tag{45}$$

These numerical estimates allow us to identify the off-diagonal $\Delta I = 2$ rescattering as the major new ingredient in the final state phases and to arrive at the result:

$$\gamma_0 = -0.1^o \; , \qquad \gamma_2 = 3.1^o \; . \tag{46}$$

We note here that the result for γ_2 is quite large, amounting to almost 50% of the strong phase δ_2 at $s = M_K^2$.

5 Sample fit to $K \to \pi\pi$ data

In this section, we provide a tentative fit to the present experimental data. This is meant as an illustration of the ideas that we have discussed above, and hopefully will provide a model for a new fully consistent experimental analysis of new data, taken with the full treatment of electromagnetic effects. We describe our treatment as tentative because it involves older data sets which were taken without the inclusion of radiative corrections. We cannot fully account for the experimental acceptances, and are forced to adopt a cruder procedure. However, the sample fit is none the less of interest because it illustrates the significant sensitivity of various quantities to electromagnetic corrections, and represents the best that can be done with the present data set.
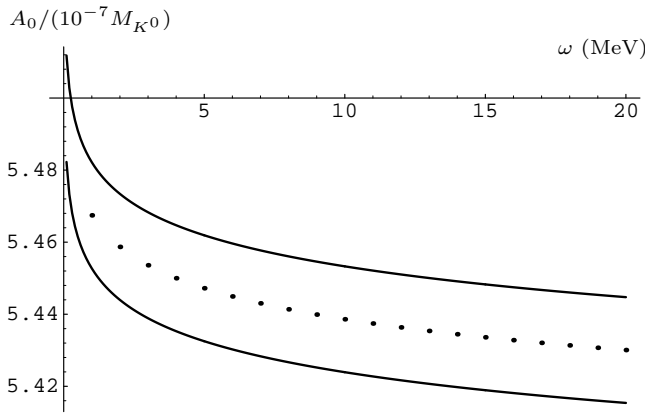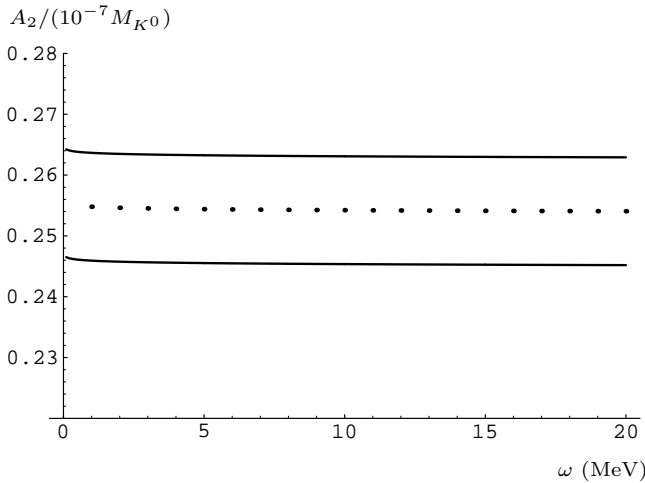
Fig. 4. Fitted A_0 as a function of ω

Fig. 5. Fitted A_2 as a function of ω
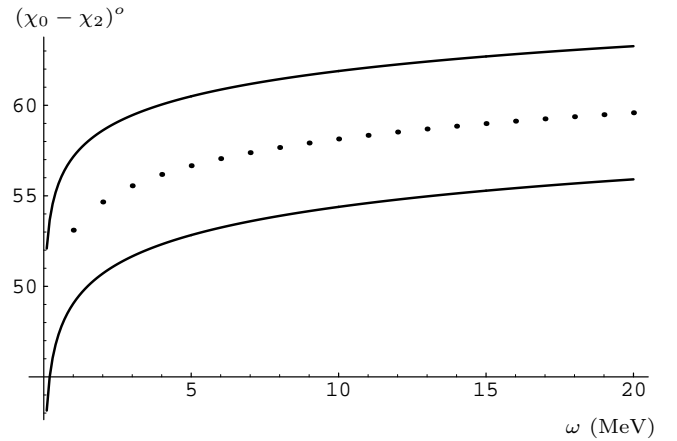

Fig. 6. Fitted $\chi_0 - \chi_2$ as a function of ω

variation of the order of 10% is seen over the considered range of ω. Thus our analysis indicates that the extraction of rescattering phases from $K \to \pi\pi$ data is sensitive to the treatment of soft photons. In the absence of precise experimental information, it is not possible to pick a definite central value for our output. We thus quote the results for the set of EM-free quantities with two error bars. The first one is due to the spread in the central values according to variations of ω between 1 and 20 MeV. The second one comes from propagating the experimental uncertainty in the decay widths and the theoretical uncertainty on the inputs δA_I^{EM}. We find:

$$A_0 = (5.450 \pm 0.020 \pm 0.015) \times 10^{-7} M_{K^0} \;,$$
$$A_2 = (0.255 \pm 0.001 \pm 0.009) \times 10^{-7} M_{K^0} \;, \quad (48)$$
$$\chi_0 - \chi_2 = (56 \pm 4 \pm 4)^o \;.$$

These results should be compared with the ones presented in Eq. (3), derived from the analysis in the isospin limit. The most important new feature is that considering EM corrections places larger error bars on all these quantities. In the case of A_2 the reason for this resides in the quite large theoretical uncertainty on δA_2^+. For A_0 and the phase difference $\chi_0 - \chi_2$, the larger error bar is due essentially to incomplete information concerning the treatment of the radiative channel. A measurement of the partial width $\Gamma_{+-}(\omega)$, with accuracy level of $\sim 0.5\%$ (this is the accuracy level of the present PDG numbers), accompanied by information on soft-photon cuts, would allow one to extract a definite central value for A_0 and $\chi_0 - \chi_2$. As a consequence, this would eliminate the first error bar associated with A_0 and $\chi_0 - \chi_2$ in (48), reducing the total uncertainty by 50% or more. Indeed, such an analysis will be performed by the KLOE experiment at DaΦne [15].

5.1 Data analysis

It will be convenient in the discussion to follow to first define

$$\chi_i \equiv \delta_i + \gamma_i \qquad (i = 0, 2) \;. \quad (47)$$

Then having $G_{+-}(\omega)$, $G_{+0}(\omega)$ and the structure dependent corrections δA_I^{EM}, one is in a position by using (13) and (10) to extract the quantities A_0, A_2 and $\chi_0 - \chi_2$.

As experimental input for the branching ratios we use the PDG averages, although these numbers come with no reference to what portion of the $K \to \pi\pi\gamma$ mode is included. In order to understand the attendant uncertainties and ambiguities of this approach, in Figs. 4, 5 and 6 we plot the output of our fit as a function of the parameter ω, the upper cutoff for IR photons in the center of mass frame. It is not clear to which value of ω (if any) the experimental numbers correspond. This ignorance gives rise to little uncertainty in the extraction A_2 and to a moderate one in the extraction of A_0, for ω varying between 1 MeV and 20 MeV[5]. However, more delicate is the situation for the extraction of the phase $\chi_0 - \chi_2$, where a

5.2 Extraction of $\delta_0 - \delta_2$: discussion

Finally we turn to the extraction of $\delta_0 - \delta_2$ from $K \to \pi\pi$ data. The relation to be used is:

$$\delta_0 - \delta_2 = (\chi_0 - \chi_2)_{\mathrm{fit}} + \gamma_2 - \gamma_0 \;. \quad (49)$$

---

[5] This range is chosen to reflect a realistic possibility for detector resolution

As shown by (49), the extraction of the strong phase difference relies on two distinct inputs:

1. The first one comes from the fit to $K \to \pi\pi$ branching ratios, which provides $\chi_0 - \chi_2$. In Sect. 5A we discussed such a fit and pointed out the sensitivity of $\chi_0 - \chi_2$ to cuts used for soft real photons. In the absence of information on these cuts a precise determination of $\chi_0 - \chi_2$ is not possible and our conclusion is that the error bars are larger than previously thought:

$$\chi_0 - \chi_2 = (56 \pm 8)^o \qquad (50)$$

2. The second input concerns the magnitude of $\gamma_2 - \gamma_0$. We have thus far established a general framework (based on unitarity) for the analysis of these isospin breaking phases. We found that γ_2 receives a $\Delta I = 1/2$ enhancement and the dominant effect is due to isospin mixing in the $\pi\pi$ rescattering rather than to radiative intermediate channels. We have provided an estimate of T_{20} at lowest order in the chiral expansion, leading us to write:

$$\gamma_2 - \gamma_0 = 3.2^o + \gamma_2^{(e^2 p^2)} \ . \qquad (51)$$

The possibility of large chiral corrections to T_{20} (associated with $\gamma_2^{(e^2 p^2)}$) cannot be ruled out, given the results obtained in the analysis of other EM corrections (violations of Dashen's theorem and the $K \to \pi\pi$ amplitudes).

In light of the previous discussion, we give the following value for $\delta_0 - \delta_2$ from $K \to \pi\pi$ data:

$$\delta_0 - \delta_2 = \left(59 + \gamma_2^{(e^2 p^2)} \pm 8\right)^o \ . \qquad (52)$$

The leading order estimate for γ_2 is seen to worsen the discrepancy between the central values of weak and strong determinations of $\delta_0 - \delta_2$. However, the large uncertainty associated with radiative corrections makes impossible a precise comparison at this stage. In this sense, the phase puzzle is alleviated, its cause being a previous underestimate of error bars. Indeed we believe that the combined effect of radiative corrections to $\chi_0 - \chi_2$ and calculation of $\gamma_2^{(e^2 p^2)}$ can fully resolve the puzzle, providing a satisfactory theoretical formulation of the problem. In fact, once a more precise extraction of $\chi_0 - \chi_2$ becomes available, (49) can be used to extract T_{20}, and thus information on the isospin breaking dynamics in $\pi\pi$ scattering at $s = M_K^2$.

6 Impact on CP phenomenology

In the present section we focus on the consequences of our work to CP phenomenology in the kaon system. Our work gives rise to interesting effects only in the theoretical analysis of ϵ'. In particular, we provide an estimate of the isospin breaking parameter Ω^{EM}, the effect of the $\Delta I = 5/2$ amplitude and the phase of ϵ'.

The analysis of direct CP-violation in $K \to \pi\pi$ proceeds exactly as in the standard case, except that now we work with the IR finite isospin amplitudes $\overline{A}_I$ and the final state interaction phases χ_I associated with them. One can then write

$$\epsilon' = -\frac{i}{\sqrt{2}} e^{i(\chi_2 - \chi_0)} \frac{\mathcal{R}e\overline{A}_2}{\mathcal{R}e\overline{A}_0} \left[\frac{\mathcal{I}m\overline{A}_0}{\mathcal{R}e\overline{A}_0} - \frac{\mathcal{I}m\overline{A}_2}{\mathcal{R}e\overline{A}_2}\right] \ . \qquad (53)$$

Defining

$$\overline{\omega} = \frac{\mathcal{R}e\overline{A}_2}{\mathcal{R}e\overline{A}_0} \ , \qquad (54)$$

and neglecting the small effect of $\delta A_0/A_0$ one arrives at

$$\epsilon' = -\frac{i}{\sqrt{2}} e^{i(\chi_2 - \chi_0)} \overline{\omega} \frac{\mathcal{I}mA_0}{\mathcal{R}eA_0} \left[1 - \frac{1}{\overline{\omega}} \frac{\mathcal{I}m\overline{A}_2}{\mathcal{I}mA_0}\right] \ . \qquad (55)$$

We recall here that in the Standard Model analysis the imaginary part of A_0 is generated by the so called gluonic penguin, while the phase of A_2 is generated by the electroweak penguin.

In order to make manifest the effects of electromagnetic corrections, we now further study (55). The first new effect is to be found in the parameter $\overline{\omega}$. It is due to the presence of the $\Delta I = 5/2$ amplitude, distinguishing $\overline{A}_2$ from $\overline{A}_2^+$ (see (10)). In the usual treatment one uses the parameter

$$\omega = \frac{\mathcal{R}e\overline{A}_2^+}{\mathcal{R}e\overline{A}_0} = \frac{1}{22.2} \ . \qquad (56)$$

However, our derivation shows that one should use $\overline{\omega}$. The two are related by:

$$\overline{\omega} = \frac{\mathcal{R}e\overline{A}_2^+}{\mathcal{R}e\overline{A}_0} \frac{\mathcal{R}e\overline{A}_2}{\mathcal{R}e\overline{A}_2^+} = \omega \left(1 + f_{5/2}\right) \ . \qquad (57)$$

The other relevant phenomenon is the leakage of the octet amplitude into $\overline{A}_2$, providing the dominant part of δA_2. This brings an extra contribution to the CP-violating phase of $\overline{A}_2$, essentially generated by the gluonic penguin and transferred to $\overline{A}_2$ via isospin breaking effects. This mechanism is usually parameterized by:

$$\Omega^{\mathrm{iso-brk}} = \frac{1}{\omega} \frac{\mathcal{I}m \, \delta A_2^{\mathrm{iso-brk}}}{\mathcal{I}mA_0} \ , \qquad (58)$$

where $\Omega^{\mathrm{iso-brk}}$ will have contributions from both electromagnetic effects (Ω^{EM}) and from strong interaction effects (Ω^{STR}) associated with $m_u \neq m_d$,

$$\Omega^{\mathrm{iso-brk}} \equiv \Omega^{\mathrm{EM}} + \Omega^{\mathrm{STR}} \ . \qquad (59)$$

The above observations lead us to write:

$$\epsilon' = -\frac{i}{\sqrt{2}} e^{i(\chi_2 - \chi_0)} \omega \frac{\mathcal{I}mA_0}{\mathcal{R}eA_0}$$
$$\times \left[1 - \frac{1}{\omega} \frac{\mathcal{I}mA_2}{\mathcal{I}mA_0} + f_{5/2} - \Omega^{\mathrm{iso-brk}}\right] \qquad (60)$$

Comparing (60) to the standard analysis (not including EM corrections), one identifies three new effects.

1. The factor $f_{5/2}$ appears: it can be obtained by inserting in (57) our previous estimates of δA_2 and δA_2^+ (see [3]). We find

$$f_{5/2}^{\mathrm{EM}} = (9.3 \pm 6.1) \cdot 10^{-2} . \qquad (61)$$

The large uncertainty reflects the one in δA_2^+. We thus find that this effect tends to increase (although slightly) the central value of ϵ'/ϵ. The authors of [10] find an opposite result because they use the "phenomenological" value of $A_{5/2}$. We believe that the phenomenological determination of $A_{5/2}$, as performed in [10], suffers from large systematic uncertainties due to neglecting IR effects and the EM phases γ_I.

2. One has to consider the electromagnetic contribution Ω^{EM}, to be added to existing estimates of Ω^{STR} due to strong isospin breaking. Again, the analysis performed in [3] enables us to get the magnitude of Ω^{EM}, since we calculated there the octet induced component of δA_2^{EM}. Thus we can write:

$$\Omega^{\mathrm{EM}} = \frac{\mathcal{R}e\, A_0}{\mathcal{R}e A_2} \cdot \frac{\mathcal{I}m\, \delta A_2}{\mathcal{I}m A_0} = \frac{\mathcal{R}e\, A_0}{\mathcal{R}e A_2} \cdot \frac{\mathcal{R}e\, \delta A_2}{\mathcal{R}e A_0} . \qquad (62)$$

Numerically we find:

$$\Omega^{\mathrm{EM}} = (6.0 \pm 2.5) \cdot 10^{-2} . \qquad (63)$$

3. One observes that the phase of ϵ'/ϵ is related to $\chi_0 - \chi_2$ and not to $\delta_0 - \delta_2$, although with the present accuracy it is hard to make a meaningful determination. We find

$$\Phi^{\epsilon'/\epsilon} = \left(\chi_2 - \chi_0 + \frac{\pi}{2}\right) - \frac{\pi}{4} = -(11 \pm 8)^o . \qquad (64)$$

The resulting effect on the real and imaginary part of ϵ'/ϵ is below the sensitivity of present kaon factories.

We conclude by observing that the individual terms $f_{5/2}$ and Ω^{EM} have a respectable size but enter in the expression for ϵ' with opposite sign. The net effect has a very small central value with a large uncertainty.

7 Conclusions

In this paper we have attempted a full phenomenological analysis of $K \to \pi\pi$ decays in the presence of electromagnetic interactions. We have provided a general parameterization of $K \to \pi\pi$ amplitudes to include the effect of isospin breaking interactions. Such a parameterization has allowed us to organize the calculation in terms of three main effects: structure dependent corrections (see [1–3]), electromagnetic infrared corrections, and isospin breaking in final state interactions. We have also studied the effect of electromagnetic corrections on the direct CP-violation parameter ϵ'.

7.1 IR effects: need for new B.R. measurements

It is well known that the calculation of IR effects requires knowledge of the experimental cuts used in treating the soft photons emitted in the $K \to \pi\pi$ decays. In Sect. 5 we have pointed out that the PDG numbers come with no information concerning the radiative channel, and this seriously compromises any attempt to properly include the radiative corrections. In the absence of experimental input, we have performed a calculation of the IR effects in a simple theoretical scheme (isotropic cut on the photon energy in the center of mass system). We have shown how this incomplete state of affairs produces uncertainties larger than previously thought in the EM-free quantities

We strongly urge that a measurement of the $K \to \pi\pi$ branching ratios be performed at one of the current high statistics kaon experiments. To be precise, it would be interesting to have a set of measurements of $\Gamma_n(\omega)$ ($n = +-, +0$) at different values of ω (the soft photon upper cutoff in the center of mass frame). This would allow anyone to apply our calculation of G_{+-} and G_{+0} in making a phenomenological analysis (as in Sect. 5). Of course, each distinct experimental procedure would require its own theoretical calculation of $G_{+-,+0}$. All such studies would be equally welcome, as long as they provide information on the inclusive sum of $\pi\pi$ and $\pi\pi\gamma$ channels. We stress that such measurements are necessary in order to fully address the impact of EM on $K \to \pi\pi$ decays.

7.2 Final state interaction phases

We have shown that isospin breaking changes the description of rescattering phases in $K \to \pi\pi$ decays, as Watson's theorem is no longer applicable. We have described this new feature within the general framework provided by the unitarity relations, pointing out that the relevant effect is of electromagnetic origin. In Sect. 4 we have set up the framework relating the extra phases $\gamma_{0,2}$ to EM effects in $\pi\pi$ scattering. Our leading order analysis finds a large effect in γ_2, equal to 50% of the strong phase δ_2. The general framework presented has the potential to fully resolve the long standing inconsistency between the strong determination of $\delta_0 - \delta_2$ at $s = M_K^2$ and the one emerging from $K \to \pi\pi$ data. At present, little can be concluded due to the large uncertainty in the phase $\chi_0 - \chi_2$ and the lack of a calculation for $\gamma_2^{(e^2 p^2)}$. The first problem will be solved by new measurements of the branching ratios (including proper information on radiative effects). The second problem depends on the theoretical ability to calculate EM corrections to $\pi\pi$ scattering at order $e^2 p^2$ in the chiral expansion.

7.3 CP phenomenology

Finally, we have analyzed the impact of electromagnetic corrections on CP phenomenology (see Sect. 6), pointing out the new features in the study of ϵ'/ϵ. The isospin breaking effects can be encoded into the factors Ω and $f_{5/2}$, and also affect the phase of ϵ'. Both $f_{5/2}$ and Ω receive contributions from strong isospin breaking and electromagnetism. We have provided an estimate for the electromagnetic effect, finding results of the order of 10%, for

these parameters. They appear with opposite sign, and thus do not produce sizeable shifts in the theoretical prediction of ϵ'/ϵ.

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